

EIGENFUNCTIONS OF THE LAPLACE-BELTRAMI OPERATOR ON HYPERBOLOIDS

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ABSTRACT. Eigenfunctions of the Laplace-Beltrami operator on a hyperboloid are studied in the spirit of the treatment of the spherical harmonics by Stein and Weiss. As a special case, a simple self-contained proof of Laplace's integral for a Legendre function is obtained.

In [SW71, Chapter IV, Section 2], Stein and Weiss described the spectral decomposition of the Laplace-Beltrami operator on the unit sphere. Their approach was to identify the eigenfunctions with homogeneous harmonic functions on Euclidean space.

In this article the eigenfunctions of the Laplace-Beltrami operator on a hyperboloid are identified with homogeneous harmonic functions (with respect to a Laplacian of type (p, q)) on an open cone. In the case treated by Stein and Weiss, Liouville's theorem implies that the degree of homogeneity must be a non-negative integer, whereas here the degree of homogeneity can be any complex number. This identification is used to compute spherical functions for $O(1, q)$, and consequently Laplace's integral formula for Legendre functions is obtained. Laplace's integral formula can also be obtained by using the residue theorem [WW27, §15.23]. Spherical functions for semisimple Lie groups in general are obtained using different methods (see, e.g., [Hel84, Chapter IV]).

Let $n = p + q$. Let $\mathbf{R}^{p,q}$ denote the space of real n -dimensional vectors equipped with the indefinite scalar product of signature (p, q) :

$$\mathbf{x} \cdot \mathbf{y} = {}^t \mathbf{x} Q \mathbf{y}$$

where Q is the diagonal matrix with p 1's followed by q (-1) 's along the diagonal. Write $|\mathbf{x}|^2$ for $\mathbf{x} \cdot \mathbf{x}$. There should be no confusion with the usual positive definite dot product and norm as they are never used in this paper.

Let $\mathbf{R}_+^{p,q}$ denote the subset of $\mathbf{R}^{p,q}$ consisting of those vectors for which $|\mathbf{x}|^2 > 0$. For $\mathbf{x} \in \mathbf{R}_+^{p,q}$, let $|\mathbf{x}|$ denote the positive square root of $|\mathbf{x}|^2$. Let $O(p, q)$ denote the group consisting of matrices such that ${}^t A Q A = Q$. Denote by $O(p, q)_0$ the connected component of the identity element of $O(p, q)$. Let $S^{p,q}$ denote the connected component of $(1, 0, \dots, 0)$ in the hyperboloid

$$\{\mathbf{x} : |\mathbf{x}| = 1, x_1 > 0\}.$$

Let ρ be any complex number. Let \mathcal{P}_ρ denote the space of all functions $f \in C^2(\mathbf{R}_+^{p,q})$ which are homogeneous of degree ρ , i.e., functions such that

$$f(\lambda \mathbf{x}) = \lambda^\rho f(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbf{R}_+^{p,q}, \lambda > 0.$$

Denote by Δ the differential operator $|\nabla|^2$ (using the indefinite dot product), where ∇ is the gradient operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right).$$

Define

$$\mathcal{H}_\rho = \{f \in \mathcal{P}_\rho : \Delta f = 0\}.$$

A function $u \in C^2(S^{p,q})$ is called a *spherical harmonic¹ of degree ρ* if u is the restriction to $S^{p,q}$ of a function in \mathcal{H}_ρ . Let H_ρ denote the space of spherical harmonics of degree ρ :

$$H_\rho = \{f|_{S^{p,q}} : f \in \mathcal{H}_\rho\}.$$

The *Laplace-Beltrami operator* $\Delta_{S^{p,q}}$ on $S^{p,q}$ is defined by

$$\Delta_{S^{p,q}} u = \Delta \tilde{u}|_{S^{p,q}},$$

where $\tilde{u} : \mathbf{R}_+^{p,q} \rightarrow \mathbf{C}$ is defined by $\tilde{u}(\mathbf{x}) = u(\mathbf{x}/|\mathbf{x}|)$ (the *degree zero homogeneous extension* of u).

Let $\mathbf{x}^\# = Q\mathbf{x}$. The following is easily verified:

Lemma 1. *Let $\mathbf{x} \in \mathbf{R}_+^{p,q}$. Then*

- (1) $\nabla|\mathbf{x}| = \mathbf{x}^\#/|\mathbf{x}|$.
- (2) $\nabla|\mathbf{x}|^\rho = \rho|\mathbf{x}|^{\rho-2}\mathbf{x}^\#$.
- (3) $|\mathbf{x}^\#| = |\mathbf{x}|$.
- (4) $\mathbf{x}^\# \cdot \nabla \tilde{u}(\mathbf{x}) = 0$ for any $u \in C^1(S^{p,q})$.
- (5) $\nabla \cdot \mathbf{x}^\# = n$.

Lemma 2. *If $u \in H_\rho$ then $\Delta_{S^{p,q}} u = -\rho(\rho + n - 2)u$.*

Proof. Since $u \in H_\rho$, $|\mathbf{x}|^\rho \tilde{u}(\mathbf{x}) \in \mathcal{H}_\rho$. Therefore (using the formulas in Lemma 1),

$$\begin{aligned} 0 &= \Delta(|\mathbf{x}|^\rho \tilde{u}(\mathbf{x})) \\ &= \nabla \cdot (\nabla(|\mathbf{x}|^\rho \tilde{u}(\mathbf{x}))) \\ &= \nabla \cdot (\rho|\mathbf{x}|^{\rho-2}\mathbf{x}^\# \tilde{u}(\mathbf{x}) + |\mathbf{x}|^\rho \nabla \tilde{u}(\mathbf{x})) \\ &= (\nabla(\rho|\mathbf{x}|^{\rho-2}\tilde{u}(\mathbf{x})) \cdot \mathbf{x}^\# + \rho|\mathbf{x}|^{\rho-2}\tilde{u}(\mathbf{x})(\nabla \cdot \mathbf{x}^\#) + |\mathbf{x}|^\rho \Delta \tilde{u}(\mathbf{x})) \\ &= \rho(\rho-2)|\mathbf{x}|^{\rho-4}\tilde{u}(\mathbf{x})|\mathbf{x}^\#|^2 + \rho|\mathbf{x}|^{\rho-2}\nabla \tilde{u}(\mathbf{x}) \cdot \mathbf{x}^\# + n\rho|\mathbf{x}|^{\rho-2}\tilde{u}(\mathbf{x}) + \Delta \tilde{u}(\mathbf{x}) \\ &= \rho(\rho-2)|\mathbf{x}|^{\rho-4}\tilde{u}(\mathbf{x})|\mathbf{x}|^2 + n\rho|\mathbf{x}|^{\rho-2}\tilde{u}(\mathbf{x}) + \Delta \tilde{u}(\mathbf{x}). \end{aligned}$$

Setting $|\mathbf{x}| = 1$ in the result of the above calculation yields

$$0 = \rho(\rho-2+n)\tilde{u}(\mathbf{x}) + \Delta \tilde{u}(\mathbf{x}),$$

from which the lemma follows. \square

¹Perhaps a more apt name would be *hyperboloidal harmonic*.

The following proposition gives a construction of spherical harmonics when $p = 1$:

Proposition 3. *Suppose $\mathbf{c} \in \mathbf{R}_+^{1,q}$ is an isotropic vector, (meaning that $|\mathbf{c}|^2 = 0$) such that $c_1 > 0$. Then $\mathbf{c} \cdot \mathbf{x} > 0$ for all $\mathbf{x} \in S^{1,q}$. Let $f(\mathbf{x}) = (\mathbf{c} \cdot \mathbf{x})^\rho$. Then $f \in \mathcal{H}_\rho$.*

Proof. The set of points where $\mathbf{c} \cdot \mathbf{x} = 0$ form a hyperplane tangential to the cone $|\mathbf{c}|^2 = 0$. For fixed \mathbf{x} , the sign of $\mathbf{c} \cdot \mathbf{x}$ can change only when \mathbf{c} crosses this hyperplane. However, the entire half-cone

$$\{\mathbf{c} : |\mathbf{c}|^2 = 0, c_1 > 0\}$$

lies on one side of the hyperplane, because the cone is quadratic. Therefore, for each $\mathbf{x} \in S^{1,q}$, it suffices to verify that $\mathbf{c} \cdot \mathbf{x} > 0$ for $\mathbf{c} = (1, 1, 0, \dots, 0)$. In this case, $\mathbf{c} \cdot \mathbf{x} = x_1 - x_2$, which is positive since $x_1 > 0$ and $x_1^2 - x_2^2 - \dots - x_n^2 = 1$, so that $x_1 > |x_i|$ for each $i > 1$.

If $g \in C^2(\mathbf{R}_+^{p,q})$ and $\phi \in C^2(\mathbf{R})$, then

$$\Delta(\phi \circ g)(\mathbf{x}) = \phi''(g(\mathbf{x}))|\nabla g(\mathbf{x})|^2 + \phi'(g(\mathbf{x}))\Delta g(\mathbf{x}).$$

Let $g(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$, then $\nabla g(\mathbf{x}) = \mathbf{c}$, so that $|\nabla g(\mathbf{x})|^2 = 0$. Since g is linear, $\Delta g(\mathbf{x}) = 0$. Therefore $\Delta f(\mathbf{x}) = 0$. \square

Let $\mathbf{e} = (1, 0, \dots, 0)$. Then $K = \text{Stab}_{O(1,q)_0}(\mathbf{e})$ is isomorphic to $SO(q)$ and is a maximal compact subgroup of $O(1, q)_0$. The action of $O(1, q)_0$ on $S^{1,q}$ is transitive, and the K -invariant spherical harmonics on $S^{1,q}$ are precisely the K -invariant spherical functions for $O(1, q)_0$. It follows from Proposition 3 that

Proposition 4. *Let \mathbf{c} be any isotropic vector in $\mathbf{R}^{1,q}$. Then*

$$\int_K (k\mathbf{c} \cdot \mathbf{x})^\rho dk$$

is a K -invariant spherical harmonic of degree ρ on $S^{1,q}$.

Since K acts transitively on the slices of $S^{1,q}$ by the hyperplanes on which the first coordinate x_1 is constant, the value of a K -invariant spherical harmonic is simply a function of x_1 , which will be denoted by $P(x_1)$. A K -invariant spherical harmonic may be viewed as a solution to an ordinary differential equation in x_1 :

Theorem 5. *Suppose that $P_\rho(x_1)$ is the value of a K -invariant spherical harmonic which is homogeneous of degree ρ . Then P_ρ is a solution to the differential equation*

$$(6) \quad (1 - x_1^2)P_\rho''(x_1) + (1 - n)x_1P_\rho'(x_1) + \rho(\rho - 2 + n)P_\rho(x_1) = 0.$$

Proof. For any $f \in C^2(S^{1,q})$ we have

$$\begin{aligned}\nabla f(x_1/|\mathbf{x}|) &= \nabla(x_1/|\mathbf{x}|)f'(x_1/|\mathbf{x}|) \\ &= \frac{(\nabla x_1)|\mathbf{x}| - x_1 \nabla |\mathbf{x}|}{|\mathbf{x}|^2} f'(x_1/|\mathbf{x}|) \\ &= \frac{\mathbf{e}|\mathbf{x}| - (x_1/|\mathbf{x}|)\mathbf{x}^\#}{|\mathbf{x}|^2} f'(x_1/|\mathbf{x}|) \\ &= u\mathbf{v},\end{aligned}$$

where $u = |\mathbf{x}|^{-3}f'(x_1/|\mathbf{x}|)$ and $\mathbf{v} = \mathbf{e}|\mathbf{x}|^2 - x_1\mathbf{x}^\#$. Since $\Delta = |\nabla|^2$,

$$(7) \quad \Delta f(x_1/|\mathbf{x}|) = (\nabla u) \cdot \mathbf{v} + u \nabla \cdot \mathbf{v}.$$

Now,

$$\begin{aligned}\nabla u &= -3|\mathbf{x}|^{-5}\mathbf{x}^\#f'(x_1/|\mathbf{x}|) + |\mathbf{x}|^{-3}\nabla(x_1/|\mathbf{x}|)f''(x_1/|\mathbf{x}|) \\ &= -3|\mathbf{x}|^{-5}\mathbf{x}^\#f'(x_1/|\mathbf{x}|) + |\mathbf{x}|^{-6}(\mathbf{e}|\mathbf{x}|^2 - x_1\mathbf{x}^\#)f''(x_1/|\mathbf{x}|).\end{aligned}$$

and

$$\nabla \cdot \mathbf{v} = \mathbf{e} \cdot \nabla |\mathbf{x}|^2 - (\mathbf{e} \cdot \mathbf{x}^\# + nx_1) = (1-n)x_1.$$

Suppose there exists a function P such that $P(x_1) = f(\mathbf{x})$ for each \mathbf{x} such that $|\mathbf{x}| = 1$. Substituting the above values of ∇u and $\nabla \cdot \mathbf{v}$ in (7) and then setting $|\mathbf{x}| = 1$ we have,

$$\begin{aligned}\Delta_{S^{1,q}} f|_{S^{1,q}}(\mathbf{x}) &= (-3\mathbf{x}^\#P'(x_1) + (\mathbf{e} - x_1\mathbf{x}^\#)P''(x_1)) \cdot (\mathbf{e} - x_1\mathbf{x}^\#) \\ &\quad + (1-n)x_1P'(x_1).\end{aligned}$$

When $|\mathbf{x}| = 1$, $|\mathbf{e} - x_1\mathbf{x}^\#|^2 = (1 - x_1^2)$ and $(\mathbf{e} - x_1\mathbf{x}^\#) \cdot \mathbf{x}^\# = 0$ so that the above equality simplifies to

$$\Delta_{S^{1,q}} f|_{S^{1,q}}(\mathbf{x}) = (1 - x_1^2)P''(x_1) + (1-n)x_1P'(x_1).$$

Combining this with Lemma 2 gives (6). \square

Corollary 8. *For $n \geq 3$, there is (up to scaling) a unique K -invariant spherical function of degree ρ given by*

$$\int_K (k\mathbf{c} \cdot \mathbf{x})^\rho dk,$$

where \mathbf{c} is any non-zero isotropic vector in $\mathbf{R}^{1,q}$.

Proof. The ordinary differential equation (6) is linear of degree 2 with a regular singular point at $x_1 = 1$. The indicial equation at this point is

$$m(m + (n-1)/2 - 1) = 0.$$

Therefore, it has (up to scaling) at most one solution defined on $[1, \infty)$. This solution is known by Proposition 4. \square

The classical integral formula due to Laplace for Legendre functions is readily derived from the preceding analysis:

Corollary 9. *Every solution of the ordinary differential equation*

$$(1 - x^2)P''(x) - 2xP'(x) + \rho(\rho + 1)P(x) = 0$$

that is defined on $[1, \infty)$ is a scalar multiple of

$$P_\rho(x) = \frac{1}{2\pi} \int_0^{2\pi} (x + \sqrt{x^2 - 1} \cos \theta)^\rho d\theta.$$

Proof. Evaluate the formula from Corollary 8 taking $q = 2$, $\mathbf{c} = (1, 0, -1)$ and $\mathbf{x} = (x, 0, \sqrt{x^2 - 1})$. \square

REFERENCES

- [Hel84] Sigurdur Helgason, *Groups and geometric analysis*, Pure and Applied Mathematics, vol. 113, Academic Press Inc., Orlando, FL, 1984, Integral geometry, invariant differential operators, and spherical functions. MR MR754767 (86c:22017)
- [SW71] Elias M. Stein and Guido Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, Princeton, N.J., 1971, Princeton Mathematical Series, No. 32. MR MR0304972 (46 #4102)
- [WW27] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, Fourth ed., Cambridge University Press, 1927.

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